Lecture 10

Last time

We defined the fundamental solution

$$E(x) = \frac{C_n}{|x|^{n-2}} \qquad n \geq 3 \tag{1}$$

where we chose $C_n = -1/(n-2)|S^{n-1}|$ and S^{n-1} represents the area of the unit sphere in $\mathbb{R}^n$.

Let $u, \phi \in C^2(\Omega) \cap C^0(\overline{\Omega})$, $\Omega \in C^1$, open. Recall our Green identities

$$\begin{aligned} \int_\Omega \vec{\bigtriangledown} u \vec{\bigtriangledown} \phi + u \Delta \phi &= \int_{\partial\Omega} u \partial_\nu \phi \qquad G\emptyset \\ \int_\Omega u \Delta \phi - \phi \Delta u &= \int_{\partial\Omega} (u \partial_\nu \phi - \phi \partial_\nu u) \qquad GII \end{aligned} \tag{2}$$

Choose $y \in \Omega$ and let $u \in C_o^2(\Omega)$. Set ϕ to be our fundamental solution $\phi(x) = E(x-y) =: E_y(x)$. We see that ϕ has a simple pole at y, hence we excises it from our domain Ω by defining $\Omega_\epsilon = \Omega \backslash \overline{B_\epsilon(y)}$ for some $\epsilon > 0$. We define ν as a unit outward normal vector to $\partial\Omega$; in precise, $\nu \in T_x^\perp \partial\Omega$ for $x \in \partial\Omega$ s.t for any positive time trajectory nearby, $x + t\nu \notin \Omega$. Define $\vec{r} = \frac{\epsilon}{\|\epsilon\|} \in T_x^\perp \partial B_\epsilon(y)$ in a similar sense as ν, pointing away from $B_\epsilon(y)$. By Green identity II

$$\int_{\Omega_\epsilon} u \underbrace{\Delta E_y}_{=0} - E_y \Delta u = \int_{\partial\Omega} (\underbrace{u}_{\substack{=0 \\ u \in C_o(\Omega)}} \partial_\nu E_y - E_y \underbrace{\partial_\nu u}_{\substack{=0 \\ u \in C_o(\Omega)}}) - \int_{\partial B_\epsilon(y)} (u \partial_r E_y - E_y \partial_r u) \tag{3}$$

$$\implies \int_{\Omega_\epsilon} E_y \Delta u = \int_{\partial B_\epsilon(y)} (u \partial_r E_y - E_y \partial_r u) \tag{4}$$

We wish to create a bound on the LHS,

$$\int_{\partial B_\epsilon(y)} |E_y \partial_r u| \leq \frac{C_n}{\epsilon^{n-2}} \cdot \overbrace{|\vec{\bigtriangledown} u(y)|}^{\substack{\partial_r u = \sum r_i \partial_i u \\ \|r\| = 1}} \cdot \underbrace{|S^{n-1}|\epsilon^{n-1}}_{m(B_\epsilon(y))} + \underbrace{C\epsilon \cdot o(1)}_{Error\ due\ y} = O(\epsilon)$$

by our choice of C_n above. Note that the $o(1)$ stems from the definition of continuity for which we have $|x - y| \leq \epsilon$ resulting in $|\vec{\bigtriangledown} u(x) - \vec{\bigtriangledown} u(y)| = o(1)$, by the continuity of $\vec{\bigtriangledown} u$ here. Now,

$$\int u \partial_r E_y = u(y) \frac{(2-n)C_n}{\epsilon^{n-1}} \cdot |S^{n-1}|\epsilon^{n-1} + o(1) = u(y) + o(1) \qquad \text{by the choice of } C_n.$$

$$\int_{B_\epsilon(y)} |E_y \Delta u| = O(\epsilon^2) \tag{5}$$

Hence

$$LHS \leq u(y) + o(1) + O(\epsilon).$$

Class notes by Ibrahim Al Balushi

Note that by the definition of Ω_ϵ

$$\int_\Omega = \int_{\Omega_\epsilon} + \int_{B_\epsilon(y)}$$

so given our result in (5) that if $\epsilon \to 0$, our integral $\int E_y \partial_r u \to 0$ over $\partial B_\epsilon(y)$, impling that the integral over Ω_ϵ converges to the integral over Ω and hence we have

$$\underbrace{\int_{\Omega_\epsilon} E_y \Delta u}_{\longrightarrow \int_\Omega} = \overbrace{\int_{\partial B_\epsilon(y)} u \partial_r E_y}^{\longrightarrow u(y)} - \underbrace{\int_{\partial B_\epsilon(y)} E_y \partial_r u}_{\longrightarrow 0} \qquad as \qquad \epsilon \to 0.$$

The domain Ω may now be generalized to $\mathbb{R}^n$ and we have for any $u \in C_o^2(\Omega)$:

$$u(y) = \int_\Omega E(x-y)\Delta u(x)\ dx, \qquad u = E * \Delta u. \tag{6}$$

$$Poisson:\ \Delta(E * f) = f \qquad (f \in C^{0,\alpha})$$

Kind of an inverse.

Greens Formula — Integral Solution for $\Delta u = 0$ Cauchy problem

Let $\phi = E_y$ in *Green identity* II. For any $u \in C^2(\overline{\Omega})$ we have the general Green's formula

$$u(y) = \int_\Omega E_y \Delta u\ dx + \int_{\partial\Omega} \underbrace{(u\partial_\nu E_y - E_y \partial_\nu u)}_{\underline{note}:\ \neq 0;\ u \notin C_0^2(\Omega)} dS_x \qquad \text{(G III)}$$

therefore supposing u is harmonic in Ω i.e satisfies $\Delta u = 0$ in Ω then for any $y \in \Omega$:

$$u(y) = \int_{\partial\Omega} (u\partial_\nu E_y - E_y \partial_\nu u)\ dS_x \tag{7}$$

represents the solution for the Cauchy problem on Ω, interms of of its Cauchy data u and $\partial_\nu u$ on $\partial\Omega$ provided it exists. Due to Dirichlet's uniqueness theorem (one presented in Lecture 9), proved that the solution for $\Delta u = 0$ is determined by values of u on $\partial\Omega$ alone - in other words; one cannot prescribe both values of u and $\partial_\nu u$ on $\partial\Omega$. $the\ Cauchy\ problem\ for\ the\ Laplace\ equation\ generally\ has\ no\ solution.$ However, this integral solution can be used to show other important properties of harmonic functions with its domain of definition.

Suppose that $\phi(x) \in C^2(\overline{\Omega})$ satisfies $\Delta\phi = 0$ on Ω. Then

$$\Phi_y(x) = E(x-y) + \phi(x)$$

defines another fundamental solution for the Laplacian with pole at $y \in \Omega$. We have

$$u(y) = \int_\Omega \Phi_y \Delta u + \int_{\partial\Omega} u\partial_\nu \Phi_y - \Phi_y \partial_\nu u, \qquad (G\ IV)$$

From GIII: Leibniz rule :$a(x) = \int b(x,y)dy$. $b \in C,\ \partial_x b \in C$.

$$\implies \partial_x a(x) = \int \partial_x b(x,y)dy$$

$K \subset \Omega$ *compact boundary well separated* :

$$\sup_K |\partial^\alpha u| \leq C(K,\alpha)\left(\sup_\Omega |u| + \sup_\Omega |\vec{\bigtriangledown} u|\right)$$

$$\implies u \in C^\infty(\Omega)$$

Mean Value Property

Another application of Green's formula is the Mean Value Property (MVP). Suppose u is harmonic on Ω, and consider a ball centered around the pole y of our fundamental solution, in precise $B_r(y) \subset \Omega$. Setting $\phi = 1$ in Greens identity I (GI) we have

$$\int_{\Omega'} \Delta u = \int_{\partial\Omega'} \partial_\nu u \implies \int\limits_{\partial B_r(y)} \partial_r u = 0$$

(noting that the $\vec{\bigtriangledown} u \cdot \vec{\bigtriangledown}\phi$ factor vanished due to our choice of ϕ).
We use Greens formula,

$$\implies u(y) = \int\limits_{\partial B_r(y)} (u\partial_r E_y - \overbrace{E_y \partial_r u}^{\substack{E_y\ on \\ \partial B_r = 0 \\ \implies \int = 0}})\ dS_x = \underbrace{\overset{\substack{invar \\ \in \partial B_r(y)}}{\partial_r E_y}} \int\limits_{\partial B_r(y)} u\ dS_x,$$

noting that for $E(x-y)|_{\partial B_r(y)}$ vanishes. Now,

$$\partial_r E_y = \frac{d}{dr}\frac{r^{2-n}}{(2-n)|S^{n-1}|} = \frac{1}{|S^{n-1}|r^{n-1}}$$

$$\implies \frac{1}{|S^{n-1}|r^{n-1}} \int\limits_{\partial B_r(y)} u\ dS_x = \frac{1}{|\partial B_r|}\int\limits_{\partial B_r(y)} u\ dS_x \tag{8}$$

$$\tag{9}$$

therefore we have,

$$u(y) = \frac{1}{|\partial B_r|}\int\limits_{\partial B_r(y)} u\ dS_x \left(= \frac{1}{|B_r|}\int\limits_{B_r(y)} u\ dx. \right)$$

where the enclosed integral simply evaluates the average of u over its domain. This essentially means *the value of a harmonic u in a closed ball at the centre equals the average of the values of u on the surface.*

Now suppose u is *subharmonic* i.e $\Delta u \geq 0$:

$$\int_{B_r(y)} E_y \Delta u \; dx \leq \int_{\partial B_r(y)} E_y \partial_r u \; dS_x$$

$$\implies u(y) \leq \int_{\partial B_r(y)} u \; dS_x = \int_{B_r(y)} u \; dx.$$

The Maximum Principles

Suppose that $\Omega \subset \mathbb{R}^n$ is an open, bounded and connected. We will first assert a weaker form of the maximum principal.

Theorem 1 (Weak Maximum Principal)**.** *Let $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ be subharmonic ($\Delta u \geq 0$) in Ω. Then*

$$\max_{\overline{\Omega}} u = \max_{\partial\Omega} u.$$

Proof. In the stronger condition where $\Delta u > 0$, the weak max principal holds trivially since the implication $\sum_k \partial_k^2 u(x) > 0$ implies that for any $x \in \Omega$, $u(x)$ cannot be a maximum. This stems from multivariable calculus which tells us if a point p is a max then $\partial_k^2 u(p) \leq 0$ for all k, hence $\Delta u \leq 0$. However, since u is continuous on a compact set $\overline{\Omega}$, u mist attain a maximum in $\overline{\Omega}$, but since the existence of a max within Ω is impossible, u attains maximum along the boundary $\partial\Omega$.
So we go back to considering the case where $\Delta u \geq 0$ subharmonic. Define $v = |x|^2$, the square modulus of $x \in \overline{\Omega}$

$$\begin{aligned} \Delta v &= \Delta |x|^2 && (10) \\ &= \sum_k \frac{\partial^2}{\partial x_k^2} |x|^2 && (11) \\ &= \sum_k 2 = 2n > 0 && (12) \end{aligned}$$

$\Delta v > 0$ in Ω. We will make use of this fact in the following manner: for any $\epsilon > 0$, $u + \epsilon v \in C^2(\Omega) \cap C^0(\overline{\Omega})$ and satisfies $\Delta(u + \epsilon v) > 0$ in Ω. We invoke the weak max on $u + \epsilon v$,

$$\max_{\overline{\Omega}} (u + \epsilon v) = \max_{\partial\Omega} (u + \epsilon v)$$

so via the triangle inequality

$$\max_{\overline{\Omega}} u + \epsilon \min_{\overline{\Omega}} v \leq \max_{\partial\Omega} u + \epsilon \max_{\partial\Omega} v$$

where we let $\epsilon \to 0$ we obtain the desired result by the compactness of $\overline{\Omega}$ and u's continuity. □

Theorem 2. *Let $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ harmonic on Ω, then*

$$\max_{\overline{\Omega}} |u| = \max_{\partial\Omega} |u|.$$

Proof. We simply make use of $\min u = -\max(-u)$. □

(This is an important result since $u = 0$ in Ω if $u = 0$ on $\partial\Omega$ which implies an improved uniqueness theorem for the Dirichlet problem: there is no need for derivatives of u on $\partial\Omega$.)

The following is a stronger max principal where we have relaxed the conditions of u's continuity on $\partial\Omega$. The logic flows due to the MVP.

Theorem 3 (Strong Maximum Principal)**.** *Suppose $u \in C^2(\Omega)$ and subharmonic in Ω. Then either u is constant or*

$$u(y) \leq \sup_{\Omega} u \qquad \forall y \in \Omega.$$

Proof. Define $M = \sup u$ and decompose Ω such that Ω_1 defines the set of points $y \in \Omega$ where $u(y) = M$, and Ω_2 where $u < M$. In precise

$$\Omega_1 = \{y \in \Omega : \ u(y) = M\}, \qquad \Omega_2 = \{y \in \Omega : \ u(y) < M\}, \qquad \Omega = \Omega_1 \cup \Omega_2 \text{ connected by assumption.}$$

The set $\{u(y) < M\}$ defines an open set hence by continuity of u the pre-image of $\{u(y) < M\}$ under u is open and is equal to Ω_2. We need to show that Ω_1 is open, for each we do, then we arrive to our conclusion. Let $y \in \Omega_1$. u is subharmonic therefore for all r sufficiently small we have by the MVP

$$u(y) \leq \frac{1}{|\partial B_r(y)|} \int\limits_{\partial B_r(y)} u(x) \ dS_x$$

$$\implies 0 \leq \int\limits_{\partial B_r(y)} u(x) \ dS_x - |\partial B_r(y)| u(y) = \int\limits_{\partial B_r(y)} (u(x) - u(y)) \ dS_x$$

$$= \int\limits_{\partial B_r(y)} (u(x) - M) \ dS_x. \leq 0. \tag{13}$$

where we used the fact $u(y) = M$ is a constant and $\int_{\partial D} dS = Area(\partial D)$. Since $u(x) - M$ is continuous and ≤ 0, it must follow that $u(x) - M = 0$ for every $x \in B_r(y)$, with r sufficiently small. Hence for every $y \in \Omega_1$ or Ω_1, there is a nbhd $B_r(y)$ of y that is completely contained in Ω_1, ie

$$\forall y \in \Omega_1 \qquad \exists r > 0 \ s.t \ B_r(y) \subset \Omega_1 \implies \Omega_1 \ open.$$

and therefore by connectedness of Ω, Ω_1 and Ω_2 cannot be disjoint concluding the principal. □

Comparison Principal

Theorem 4. *Suppose $u, v \in C^2(\Omega) \cap C^0(\overline{\Omega})$. $\Delta u \geq \Delta v$ in Ω, $u(x) \leq v(x)$ $x \in \partial\Omega$.*

$$\implies u \leq v \qquad in\ \Omega.$$

Proof. $= u - v. \qquad w \leq 0$ on $\partial\Omega$.

$$\implies w \leq 0 \qquad in\ \Omega$$

□

Applications: Uniqueness for Dirichlet. $\Delta u = f$ in Ω, $u = g$ on $\partial\Omega$
$\Delta u \geq 0$ in Ω, $u = 0$ on $\partial\Omega$

$$\implies u \leq 0.$$

Example: $\Delta u = K u^{\alpha}$ $(K > 0,\ \alpha > 0)$ does not have a positive solution.
$u = v$ on $\partial\Omega$

$$-\Delta u \leq -\Delta v \implies u \leq v$$

$$f \leq g \implies (\Delta)^{-1} f \leq (-\Delta)^{-1} g$$